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Every Subgroup is Locally Subnormal

Selami Ercan

Gazi Üniversitesi, Gazi Eğitim Fakültesi, Matematik Eğitimi Anabilim Dali

Corresponding author: Selami Ercan

ABSTRACT: In this article we study every subgroups is locally subnormal. We prove that if G be a torsion-free group in $\wp_2 \cap \wp_n$, for all $1 \le n \in \mathbb{N}$ and assume that for every homomorphic image $\overline{G}$ of G has a non-trivial abelian normal subgroup, then G is soluble. We also prove that if G be a hyperabelian torsion-free group in $\wp_{2,n}$, then G is nilpotent group.

Keywords: locally subnormal,hyperabelian, soluble, Engel group.

INTRODUCTION

Subnormal Subgroups:

A group H of the group G is said to be subnormal if H is a term of a finite series of G; if there exists $d \in \mathbb{N}$ and series of subgroups, such that

$$H = H_d \lhd H_{d-1} \lhd ... \lhd H_0 = G$$

If $H \lhd\lhd G$ then the defect of H in G is the shortest lenght of such a series : it will be denoted by d(H,G). We shall saythat a subgroup H of G is n-subnormal if $H \lhd\lhd G$ and d(H,G)≤n. We denote by $\wp_1$ the class of all groups which every subgroup H of G is subnormal.

Normalize Condition:

A group G said to satisfy the normalizer condition if
$H \ne N_G(H)$ for all proper subgroups H of G. We denote by $\wp$ the class of all groups satisfying normalizer condition.

Locally Subnormality

A class which in intermediate between $\wp_1$ and $\wp$ class of groups in which every subgroup is locally subnormal; where a subgroup H of a group G is called locally subnormal if
$H \lhd\lhd \langle H, X \rangle$ for all finite $X \subset G$. The class we denoted by $\wp_2$.

Isolator:

Let H be a subgroup of a group G. the isolator of H in G is the set

$$I_G(H) = \{ x \in G : x^n \in H \, , \, for \;\; some \; 1 \le n \in \mathbb{N} \}$$

Let us denote by $\wp_n$ the class groups in which every soluble subgroups is soluble of derived length n,for each $1 \le n \in \mathbb{N}$

$\wp_{2,n}$ class of groups in which every subgroup is locally subnormal; where a subgroup H of a group G is called locally subnormal if $H \lhd\lhd G$ for all finite $X \subset G$ and subnormal indices bounded n,where $1 \le n \in \mathbb{N}$.

Mainresults

Lemma 1.

Let G be a group in $\wp_2$. Then G is locally nilpotent.

Proof.

Let F be finitely generated subgroup of G. By hypothesis, for all $x \in F$, $\langle x \rangle$ is subnormal subgroup of F. F is a nilpotent group by 12.2.8(1). Thus G is locally nilpotent.

Lemma 2.

Let G be a hyperabelian torsion-free group in $\wp_2 \cap \wp_n$, for all $1 \le n \in \mathbb{N}$. Then G is soluble.

Proof.

Suppose that G is not soluble. Since G ishyperabelian, then there exists non-soluble normal subgroup L of G such that $L = \bigcup_{i \in \mathbb{N}} A_i$, where A_i is soluble normal subgroups of G, for all $i \in \mathbb{N}$ by (5)Lemma (3). But by hypothesis, for all $i \in \mathbb{N}$, A_i is soluble of derived length n. Thus L is soluble. This is a contradiction.

Theorem A .

Let G be a torsion-free group in $\wp_2 \cap \wp_n$, for all $1 \le n \in \mathbb{N}$ and assume that for every homomorphic image $\overline{G}$ of G has a non-trivial abelian normal subgroup. Then G is soluble.

Proof.

Suppose that G is not soluble. By hypothesis G has an abelian normal subgroup A of G. $I_G(A)$ is an abelian normal subgroup of G by (1)Lemma (ii),(v)(3). If $G = I_G(A)$, then G is soluble by (1)(v) (3). Assume that $G \ne I_G(A) = A_1$.

G/A_1 is torsion-free. By hypothesis, G/A_1 has an abelian group B_1/A_1. Also B_1 is soluble. If $I_G(B_1) = G$, then G is soluble. Assume that $A_2 = I_G(B_1) \ne G$. Then G/A_2 is torsion-free. Now assume that we have subgroups $A_\lambda \ne G, A_\lambda \lhd G$ and G/A_λ is torsion-free, α is ordinal number, for all $\lambda \le \alpha$, α>1. If α is limit ordinal, $A_\alpha = \bigcup_{\lambda \le \alpha} A_\lambda$.

If α is not limit ordinal,there exists . $\alpha - 1$ such that $G/A_{\alpha-1}$ is torsion-free. Similar to the definition of $A_\alpha/A_{\alpha-1}$ is defined in $G/A_{\alpha-1}$ and $G/A_{\alpha-1}$ is torsion-free. If it is continued in such a way that there exists a ordinal number β such that $G = A_\beta$. $\{A_\alpha : \alpha \le \beta\}$ is hyperabelian series of G. Thus G is soluble by Lemma 2.

Lemma 3.

Let G be a group in $\wp_2$. Then G is an Engel group.

Proof.

For all $x, y \in G$, by hypothesis $\langle x \rangle \lhd\lhd \langle x, y \rangle$.Then there exists a n natural numbers such that $\left[\langle x, y \rangle, n\langle x \rangle\right] \le \langle x \rangle$

This implies that

$$[y,(n+1)x]=1$$

Thus G is Engel group.

Corollary

Let G be a hyperabelian torsion-free group in $\wp_{2,n}$. Then G is nilpotent group.

Proof.

G is soluble by Teorem A. For all

$x, y \in G$

$\langle x\rangle$ is subnormal in $\langle x, y\rangle$ and $\langle y\rangle$ is subnormal in $\langle x, y\rangle$. Since subnormal indices is n, $ $[\langle x, y\rangle, n\langle x\rangle] \leq \langle x\rangle$ and $[\langle x, y\rangle, n\langle y\rangle] \leq \langle y\rangle$. This implies that $[y,(n+1)x]=1$ and $[x,(n+1)y]=1$. Thus G is (n+1)-Engel group. G is nilpotent by Corollary in page 64(2).

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